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Influence of Hoek-Brown linearization and input cross-correlation in slope Probability of Failure Calculation

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Abstract
When Hoek-Brown failure-criterion (H-B) is selected, their parameters are determined as the ones that better fit data; therefore, depends on the method of fitting, the experimental error and statistical sufficiency of data. If these uncertainties are considered, H-B parameters are random variables. On the other hand, in slope stability analyses, is common to linearize the H-B criterion to estimate equivalent friction angle, ϕ_e, and cohesion, c_e, which generate a non-lineal relation between ϕ_e and c_e. In this paper, we examine the effect of this linearization in probabilistic analyses that use a Mohr-Coulomb criterion when the input data are random H-B parameters. We study the uncertainty propagation from H-B parameters to ϕ_e and c_e. Additionally, because linearization cross-correlate ϕ_e and c_e, we analyze the effect of the correlation $\rho_{\phi c}$ in Factor of Safety (FS) and Probability of Failure (PoF) results, finally because also H-B parameters can be correlated we examine the effects on FS and PoF.

1. Introduction

Properties of geotechnical materials are affected by various factors during their formation process, such as properties of parent materials, magmatism, sedimentation, weathering, erosion processes, and therefore, vary spatially. Additionally, several uncertainties are incorporated in the estimation of the properties as result of imperfect test equipment and procedural errors, statistical uncertainty arising from insufficient number of tests, and the transformation uncertainty related to the interpretation of results (Cao et al., 2017). For example, if Hoek-Brown rock mass failure criterion is adopted, there is error and variability associated to the results of triaxial and unconfined compressive tests, and the number of available tests is commonly restricted. Also, the values of the parameters depend on the method of fitting (e.g. non-linear least squares or Bayesian sampling, absolute residual or relative residual, etc.)

Probability theory and statistics gives a framework to rationally incorporate these uncertainties into geotechnical analysis, allowing to measure the performance of geotechnical structures statistically, for example, through the reliability index, β, and the probability of failure, PoF. The last one, defined as the probability of performance requirements not being satisfied, has grown in use as an acceptance criterion because combined with risk assessment is a good option to deal with uncertainty and variability, and several methods have been proposed to estimate it, such as the first-order second-moment method (Christian et al., 1974; Hassan & Wolff, 1999), the first-order reliability method (Low et al., 1988; Low, 2003), response surface methods (Morgan & Henrion, 1990 and Tandjiria at al., 2000) and direct Monte Carlo simulation method (Griffiths & Fenton, 2004; El Ramly, 2005). The response surface method has been used in risk-based slope design applications as described by Steffen *et al.* (2008) and Contreras (2015). This approach has the advantage of combining the rigor of a Monte Carlo simulation with the practicality of requiring fewer FS calculations with the geotechnical model to construct the response surface used as a surrogate model in the process.

On the other hand, for rock masses is generally accepted to use non-linear Hoek-Brown (H-B) failure envelopes. Hoek (2007) recommends, where possible, to apply the criterion directly; however, given that many geotechnical design calculations are written for the Mohr-Coulomb failure criterion (M-C), it is often necessary to calculate equivalent rock mass cohesion, c_e, and friction, ϕ_e (Eberhardt, 2012). This is particularly true in rock slope analyses, for example, to include the weakness induced by joints, it is

common to use a linear weighing of the strength contributed by the rock mass and joints fractions of material (e.g. Jennings, 1970); therefore, linear failure envelopes, as the Mohr-Coulomb criterion, are used to represent equivalent strength.

The linearization of H-B criterion is done by fitting an average linear relationship to the H-B curve in the range $\sigma_t < \sigma_3 < \sigma_{3max}$, process which involves balancing the areas above and below the M-C plot (Hoek et al., 2002). The output of this process are explicit expressions for ϕ_e and c_e in terms of H-B parameters (a, s, m_b, σ_{ci}, GSI, D) and σ_{3max}. As a result, the linearization of H-B parameters implies a non-lineal relation between ϕ_e and c_e.

If H-B parameters are considered as random variables, the linearization operation propagates their uncertainties (intrinsic to their distributions), generating random variables Φ (friction) and C (cohesion) for which we do not know their distributions a priori, but we know that are correlated.

However, because lack of data, it is common to assume that strength properties are independent random variables. If a Mohr-Coulomb (M-C) failure criterion is used to model the material failure envelope, this means that friction angle (ϕ) and cohesion (c) are considered independent; however, cross-correlation between ϕ and c has been widely reported (e.g. Young, 1986; Di Matteo et al., 2013), and ignoring their cross-correlation may lead to the underestimation or overestimation of the failure probability (e.g. Jiang et al., 2014; Wu, 2013; Johari & Mehrabani Lari, 2017). Other option is to use linear correlations between ϕ-c, which can be obtained through laboratory tests, but normally the available information is not enough to statistically define this coefficient.

Furthermore, the H-B random variables can also be correlated to each other, as for example the reasonable correlation between the ratio σ_{ci}/σ_t and the parameter m_i (Sheorey, 1997); therefore, in the same sense that for M-C criterion, neglecting cross-correlation between H-B parameters could generate underestimation or overestimation of PoF.

For this reason, in this paper we study the uncertainty propagation from Hoek-Brown parameters to ϕ_e and c_e, i.e. we study the distributions of random variables Φ (friction) and C (cohesion) obtained from the H-B linearization; the influence of ϕ_e-c_e cross-correlation induced by linearization in the FS and PoF, and finally, we study the effect of the correlation of H-B parameters in FS and PoF.

This paper is organized as follow: Section 2 outlines the methodology used along this paper. Section 3 shows the obtained results and the analysis of them, and finally, in Section 4 the conclusions of this work are presented.

2. Methodology

The scheme of the work of this paper is shown in Figure 1. The first step consists in to generate random variables for friction Φ and cohesion C, that comes from linearization of Hoek-Brown parameters, which are in turn, correlated random variables. For simplicity, we select m_i and σ_{ci} as random variables (M and S), and the other properties which not depend on them (GSI, D) are considered as constants.

Uniform and normal probability density functions (pdf) were considered for M and S. The former one because is the pdf with largest entropy, i.e. minimizes the information assumed into the distribution, while the later one because is the commonly found distribution for data. M and S distributions are generated with specific correlations ρ_{MS}, imposed by means of Gaussian copulas. Linearization of M and S generate distributions of Φ and C. At this point, we study the effect of ρ_{MS} over the distribution of Φ and C.

Pairs of discrete values of ϕ_e and c_e, with correlation coefficient $\rho_{\phi c}$ induced by the linearization, are used to calculate discrete values of FS. By doing this a large number of times, we can study the distribution of FS and obtain the PoF as the number of simulations with FS<1 divided by total number of simulations. Additionally, we make Monte Carlo simulations over Φ and C to obtain pairs (ϕ_e, c_e) with different correlation coefficients $\rho_{\phi c}$. With this new re-sampling we calculate FS and study the influence of $\rho_{\phi c}$ in PoF. Finally, we change the values of ρ_{MS} to observe its influence in FS and PoF.

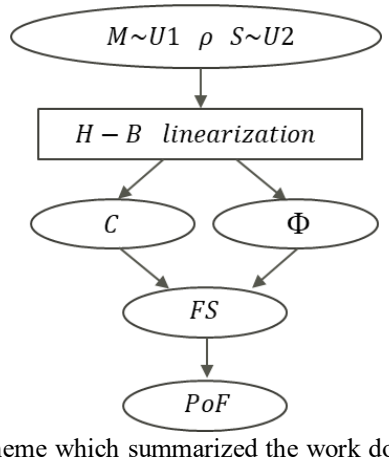

Figure 1. Scheme which summarized the work done in this paper. From correlated H-B properties, M-C properties are obtained and the effects on FS and PoF are studied.

Because PoF evaluation implies the calculation of thousands of FS computations, FS were calculated by means of the Equation (1) (Carranza-Torres & Hormazábal, 2018), allowing a quick calculation of PoF. Additionally, to have an explicit expression for the Factor of Safety can be advantageous during the analysis of the results:

$$\frac{FS}{\tan\emptyset} = \frac{1}{\tan\alpha} + \frac{g_1(\alpha)}{Xad} + \frac{g_2(\alpha)}{Xad^{g_3(\alpha)}} \quad (1)$$

Where X_{ad} is the dimensionless parameter given by Equation (2), and $g_i(\alpha)$ are cubic polynomials of the slope angle given in detail in Carranza-Torres & Hormazábal (2018).

$$Xad = \frac{\gamma H \tan\emptyset}{c} \quad (2)$$

Hence, Equation (1) depends on cohesion (c), friction angle (ϕ), slope height (H), unit weight (γ) and slope angle (α), and is valid for slopes of arbitrary height and inclination in homogeneous/isotropic dry ground, that obeys the Mohr-Coulomb shear failure criterion.

3. Results

3.1 Effect of linearization to C and Φ

The objective of this section is to obtain an idea of how variability of m_i and σ_{ci}, represented by the random variables M and S, is propagated by the linearization in the random variables C and Φ:

$$L(M,S) \rightarrow (C,\varphi) \quad (3)$$

Where L represents linearization operation. Figure 2 shows how (m_i, σ_{ci}) points obtained from uniform (a right side) and normal truncated (at left side) distributions are mapped into ϕ_e - c_e space by L. Red colors identify zones with more density of points. The spatial bounds of ϕ_e - c_e are clearly non-linear, and there is a strong loss of symmetry in their distribution; therefore, the obtained probability density functions for C and Φ have skewness, as shown in Figure 3, where it is possible to see a normalized histogram of ϕ_e obtained from uniform distributed M and S.

Several runs were done for different ranges of m_i, σ_{ci}, where minimum, maximum and mean distribution values were selected randomly. We observe the behavior of the obtained distributions in terms of their 'shape', and their first, second and third moments. We made the following general conclusions:

- There is a great variability in the resulting distributions C and Φ depending for example in the range of input values m_i, σ_{ci}. Hence, is not possible (the option of an analytical calculation was not considered in this study) to indicate a close expression for the resulting distributions and their parameters.
- As expected (see Eberhardt, 2012), sample means of M have a positive correlation with sample means of Φ, while S is correlated with C. M and C appears uncorrelated, same that S and Φ. The effect of correlation ρ_{MS} is negligible.
- The same as above was observed for variances. It was also observed that when M variance increase, the statistical dispersion of Φ is also increased (Figure 2). The same for S and C.
- C and Φ are truncated pdf. This behavior is expected because M and S are also truncated pdf.
- In general, the shape of the obtained pdfs, C and Φ, are similar to the 'root' pdfs, M and S, i.e. if M and S distributed as uniform, then C and Φ distribute as uniform; however, as variance of M and S increase, skewness also increases in C and Φ, where C is more prone to suffer positive skewness, while Φ negative skewness (see Figure 4).
- When $\rho_{MS} = 0$, C and Φ are more prone to distribute as normal.

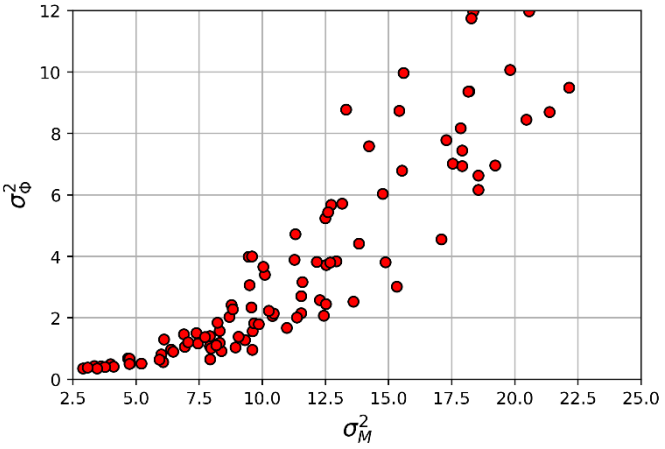

Figure 2. Comparison of variances of M and Φ. It is observed that variance of Φ(output) increase when variance of M (input) increase. The statistical dispersion is also increased when variance of M increase.

Therefore, if variabilities of m_i, σ_{ci} are non-negligible, it is necessary to incorporate their effect in

probabilistic analyses, calculating their means, variances, verify how correct is the use of symmetric distributions of C and Φ, etc. A Monte Carlo simulation appears as a simple method to study the statistical characteristics of C and Φ.

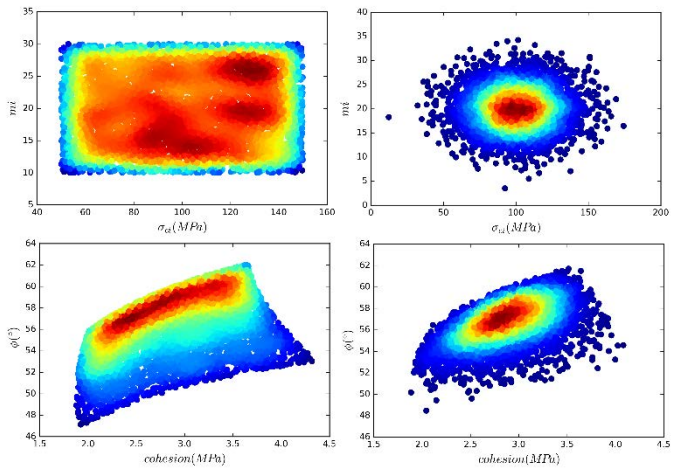

Figure 3. Scatter plots of discrete points of (m_i, σ_{ci}) and (ϕ_e, c_e) where red color represent more density. At the upper left side, (m_i, σ_{ci}) points obtained from uniform distribution and below him, the corresponding points (ϕ_e, c_e). At the right side, (m_i, σ_{ci}) points obtained from normal distribution and the corresponding (ϕ_e, c_e) points.

It is important to mention that pairs (ϕ_e, c_e) obtained from linearization of H-B, and used to construct C and Φ, are correlated and follows a non-linear relation. We can be tempted to use these probability distributions of C and Φ, like the one in Figure 4, and calculate PoF without considering their correlation $\rho_{\phi c}$ (e.g. by a MonteCarlo simulation), but this procedure could generate an error in PoF calculation. In fact, the use of the linear Pearson correlation coefficient could be inadequate to characterize the correlation between ϕ_e and c_e, due to a marked non-linear behavior, and a different correlation coefficient, as the distance correlation (Szekely et al., 2007), could be better to represent how c and ϕ are related.

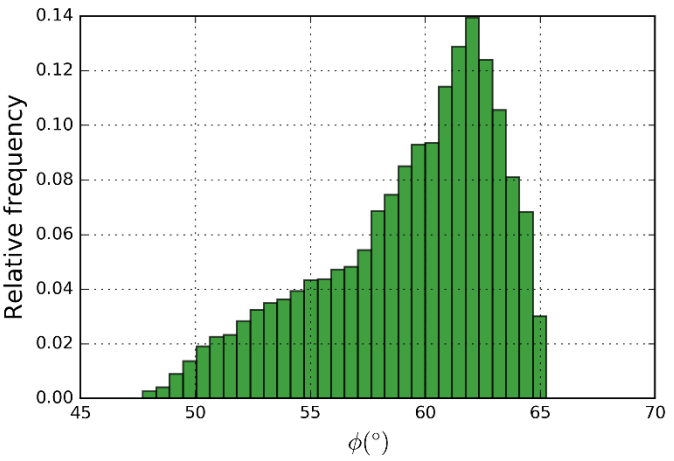

Figure 4. Normalized histogram of ϕ_e, representing Φ distribution, that was obtained from uniform distributions of M and S, showing positive skewness.

3.2 Effect of ρ_{MS} on PoF

Because m_i and σ_{ci} are the variables of interest, fixed values of H=250m, β=55°, γ=27kN/m^3, and r_u=0.5 were considered in Equation (1). On the other hand, to calculate the equivalent friction and cohesion, a GSI=40 and D=0.7 were used (the epistemic uncertainty in these last parameters was not considered in this study, because it is focused in the variables with natural variability (Baecher & Christian, 2003), but their incorporation is an ongoing work). The FS function obtained with Equation (1) is shown as contour plot in Figure 5.

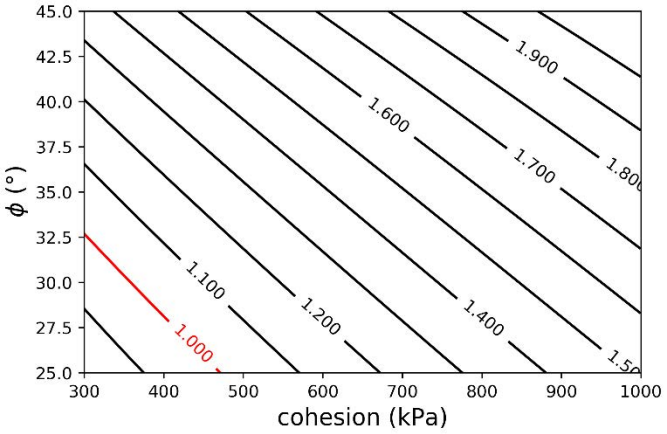

Figure 5. Contour plot representation of the Factor of Safety obtained with the Equation (1) (Carranza-Torres and Hormazabal, 2018) in terms of cohesion and friction. Red line shows the isocontour of FS=1.

Clearly, it is non-common to have an explicit representation for the FS as Figure 5; however, we can take advantage of this to better understand the influence of ρ_{MS} in PoF. Figure 6 shows the location of 4,000 pairs (c, ϕ), where the red ones were obtained from linearization, and the blue ones represent uncorrelated data ($\rho_{\phi c}$=0).

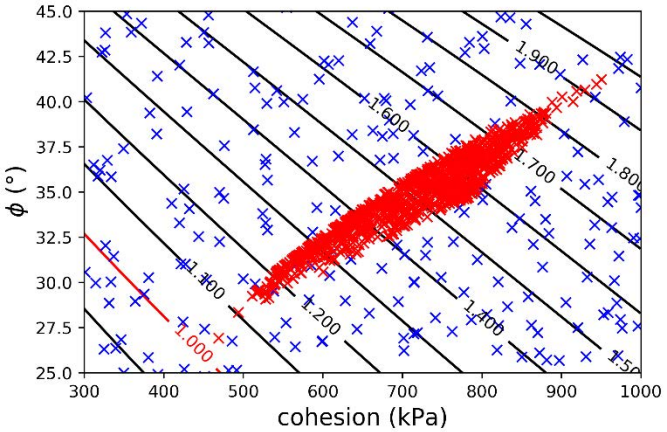

Figure 6. Random pairs (c,ϕ) over the contour representation of FS. In blue, uncorrelated points, while red points represent correlation points obtained from H-B linearization.

For uncorrelated data ($\rho_{\phi c}$=0), all the pairs (c,ϕ) in range Rc × Rϕ (where Ri =[i_{min},i_{max}]) have the same probability to occur; therefore, the PoF can be interpreted as the ratio between the area in which FS<1 and the total area Rc × Rϕ:

$$\mathrm{PoF}(\rho_{c\varphi}=0) \approx \frac{\int_{Rc}\int_{R\varphi} H(-FS(c,\varphi))\,dc\,d\varphi}{Rc\,R\varphi} \quad (4)$$

Where H(·) is the Heaviside function. On the other hand, Figure 6 shows that linearization restricts the area of feasible pairs (c,ϕ). In this case PoF could be calculated similarly to the former case as:

$$\mathrm{PoF(lin)} \approx \left(\int_{\omega} H(-\mathrm{FS}(c,\varphi))d\omega\right)/\left(\int_{\omega} d\omega\right) \quad (5)$$

Where ω is the region of feasible pairs (c_e, ϕ_e). Therefore, the correlation between M and S, ρ_{MS}, modifies at first the set of possible values for m_i and σ_{ci}. Figure 7 shows in scatter plots, the sets of possible values of m_i and σ_{ci} obtained for a negative correlation ρ_{MS} =-0.9.

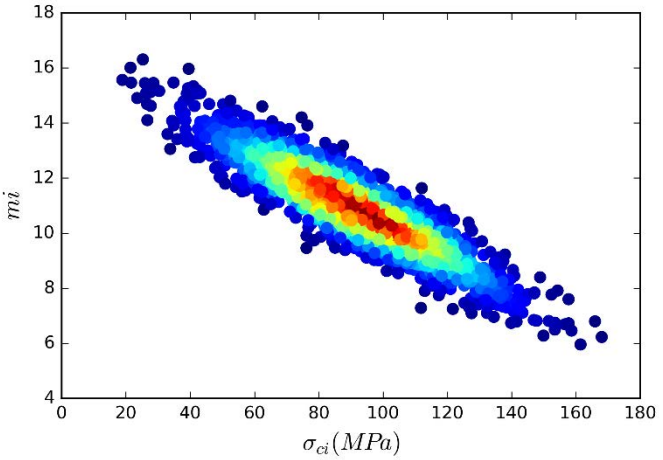

Figure 7. Scatter plots of the set of possible values of m_i and σ_{ci} for ρ_{MS}=-1.

This effect impact directly in the set of feasible values of pairs (c, ϕ), and therefore, in the results of PoF. To evaluate the effect of the correlation Figure 8 shows PoF results obtained for different values of correlation ρ_{MS}. It is observed that for this case, PoF increase when the correlation is positive, and decrease when correlation is negative, existing a difference of 8% between minimum and maximum values of PoF. In the case ρ_{MS}=-1 the region of feasible values of c and ϕ, must be near to parallel to the iso-contours of FS (see Figure 5) and to be above of the iso-contour FS=1, which explains the PoF near to zero. On the other hand, the region of feasible points ρ_{MS}=1 is near to perpendicular to iso contours, which explains that the maximum values of PoF are encounter for this value of ρ_{MS}; however, this behavior cannot be generalized, and each case needs to be analyzed separately.

These results indicate that this correlation is an important input in probabilistic studies and needs to be considered, and a possible way to do it is by means of Bayesian methods as in Contreras et al. (2016).

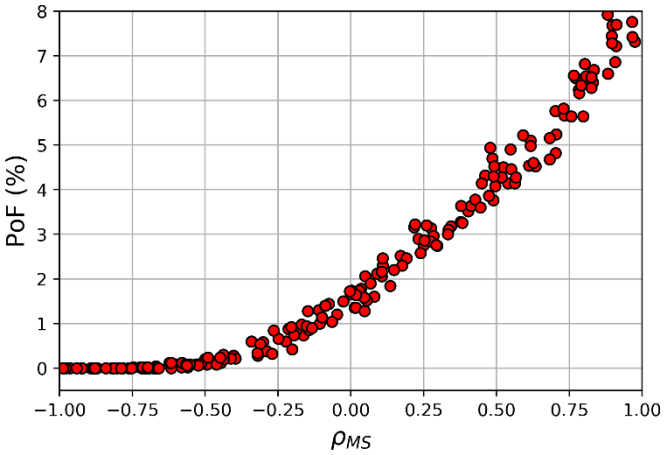

Figure 8. Effect of ρ_{MS} on PoF.

4. Conclusions

In this paper, we analyze the effect of the linearization of Hoek-Brown criterion commonly done in slope stability, into Factor of Safety and Probability of Failure calculations, considering parameters m_i and σ_{ci} as random variables.

First, we observe that H-B linearization generates skewness in the distributions of c and ϕ, and therefore, considering symmetric distributions C and Φ could induce inaccuracies in results.

The correlation induced by H-B linearization needs to be considered, because PoF results depend greatly of the correlation of c, ϕ . Therefore, pairs of points (c, ϕ) generated during linearization must be used for FS calculations, and no pairs (c, ϕ) randomly sampled from pdf's C and Φ, for example by Monte Carlo.

Depending on the range of m_i and σ_{ci}, the correlation coefficient of c and ϕ after linearization varies from -1 to 1. Sometimes, non-linear relation is found between c and ϕ, which indicate that it is better to use a correlation measurement different that the classical Pearson correlation (linear).

Finally, it was observed that the correlation between the inputs H-B parameters also has an important effect in PoF results. This correlation is difficult to measure, because depends, for example, in the method of fitting of experimental points, but if exists can have a non-negligible effect in results.

It is important to mention that GSI and D parameters, which have an important effect in equivalent cohesion and friction were not considered in this study, but their influence is an ongoing work.

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